SYDNEY GRAMMAR SCHOOL

2012 Trial Examination

FORM VI
MATHEMATICS EXTENSION 2

Monday 6th August 2012

General Instructions

- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet and on the tear-off sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 85 boys

Examiner

KWM/BDD

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The value of $\displaystyle\int_0^{\pi} 5\sin x\cos^4 x\,dx$ is:

(A) 0 (B) 2 (C) -2 (D) 20

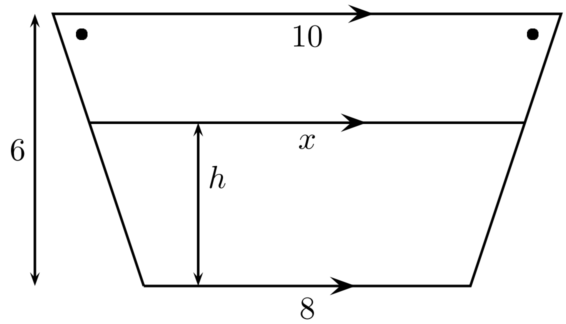
QUESTION TWO

The eccentricity of the ellipse $\dfrac{x^2}{100}+\dfrac{y^2}{36}=1$ is $e=\dfrac{4}{5}$.

The distance between the two foci is:

(A) 8 (B) 16 (C) 20 (D) 25

QUESTION THREE



The diagram above shows a trapezium with an interval x units in length drawn parallel to the base and h units from the base. An expression for x in terms of h is given by:

(A) $x=8+\frac{h}{6}$

(B) $x=12+\frac{h}{8}$

(C) $x=8+\frac{h}{3}$

(D) $x=5-\frac{6}{h}$

Exam continues next page ...

QUESTION FOUR

A motor bike and rider with mass 200 kg accelerates under a propelling force of 12 800 Newtons and as it moves it experiences a resisting force of $2v^2$ Newtons, where v m/s is the velocity. The motion is therefore described by the equation $\ddot{x} = 64 - \dfrac{v^2}{100}$.

What is the maximum speed attained by the bike?

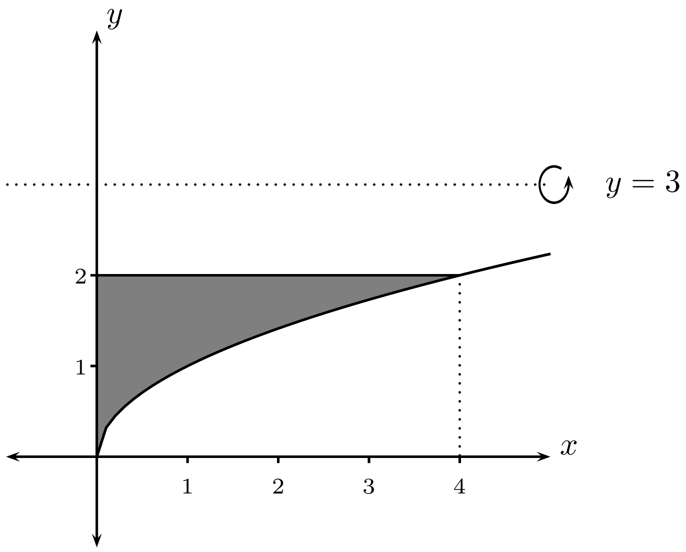
(A) 288 km/h

(B) 80 km/h

(C) 280 km/h

(D) $\sqrt{32}$ m/s

QUESTION FIVE



The area bounded by the curve $y = \sqrt{x}$, the y-axis and the line $y = 2$ is rotated about the line $y = 3$. The volume is to be calculated by taking slices perpendicular to the axis of rotation. Which integral gives the volume of the solid formed?

(A) $\pi \displaystyle\int_0^2 \Big((3-y)^2 - 1\Big)\, dy$

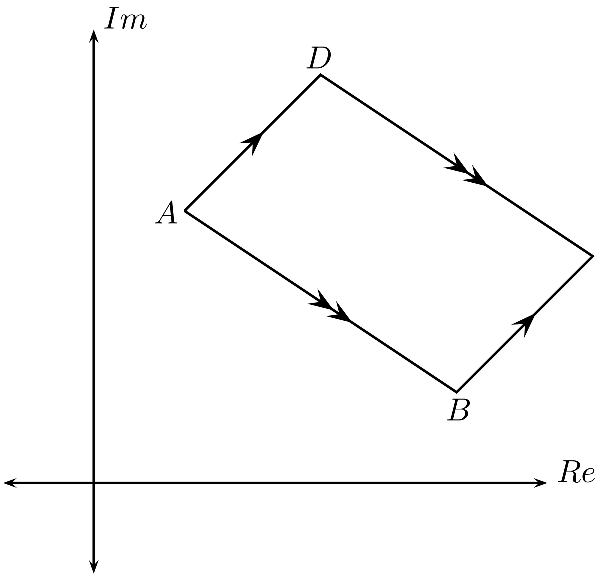
(B) $\pi \displaystyle\int_0^4 \Big(x - 6\sqrt{x} + 8\Big)\, dx$

(C) $\pi \displaystyle\int_0^4 \Big(2 - 2\sqrt{x} + x\Big)\, dx$

(D) $\pi \displaystyle\int_0^4 \Big((3-x)^2 - 1\Big)\, dx$

Exam continues overleaf ...

QUESTION SIX



The diagram above shows parallelogram $ABCD$ drawn in the first quadrant of the complex plane. The points A, B and C represent the complex numbers z_1, z_2 and z_3 respectively. The vector DB represents the complex number:

(A) $z_1 + z_3 - 2z_2$

(B) $z_2 - z_1 - z_3$

(C) $2z_2 - z_1 - z_3$

(D) $2z_2 - z_1 + z_3$

QUESTION SEVEN

The complex number ω is a root of the equation $z^3 + 1 = 0$.
Which of the following is FALSE?

(A) $\overline{\omega}$ is also a root.

(B) $\omega^2 + 1 - \omega = 0$

(C) $\frac{1}{\omega}$ is also a root.

(D) $(\omega - 1)^3 = -1$

Exam continues next page ...

QUESTION EIGHT

The value of $\lim_{N \to \infty} \int_0^N e^{-x}\,dx$ is:

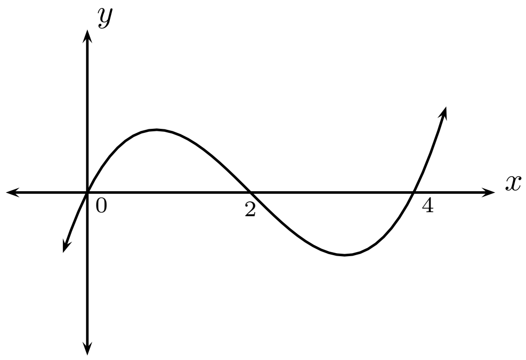
(A) 0

(B) 1

(C) -1

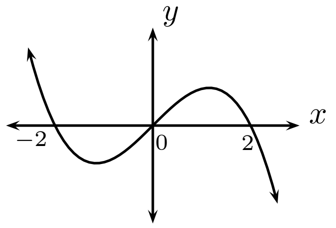
(D) ∞

QUESTION NINE

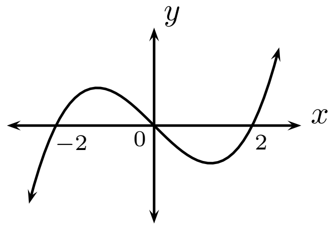


The graph of $y = f(x)$ is shown above. The graph of $y = f(2 - x)$ is:

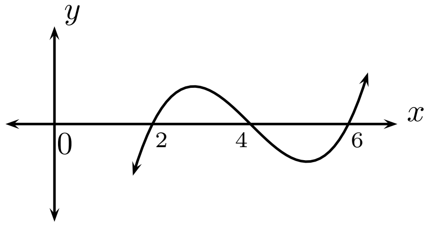
(A)



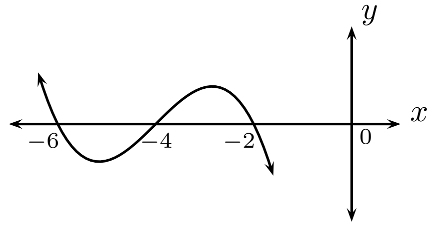
(B)



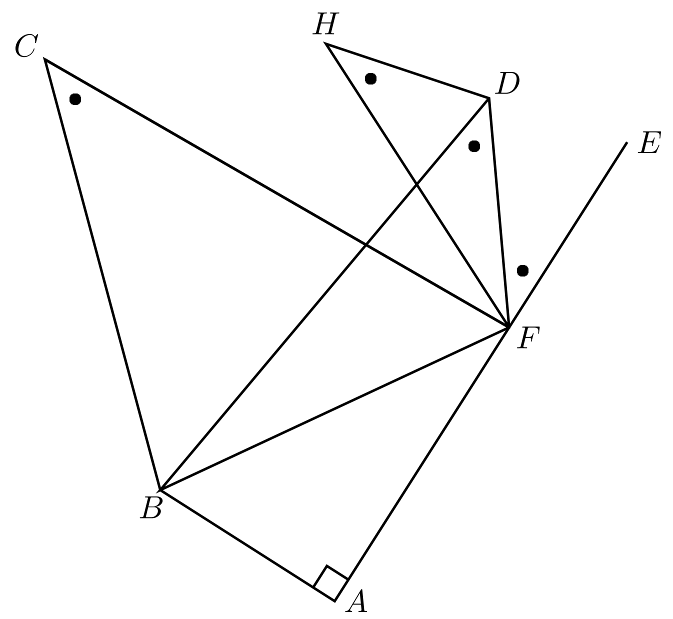
(C)



(D)



QUESTION TEN



In the diagram above, all equal angles are marked with a dot and AFE is straight. Which of these statements is INCORRECT?

(A) A circle may be drawn through A, B, F with diameter BF

(B) The points B, C, D, F are concyclic.

(C) A circle may be drawn through D, F, H with tangent AE

(D) A circle may be drawn through B, D, F with tangent AE.

End of Section I

Exam continues next page . . .

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\int x\cos x\,dx$. **1**

(b) Find $\displaystyle\int \frac{x+1}{x-2}\,dx$. **2**

(c) Use the substitution $x = 2\cos\theta$ to evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx\,.$ **3**

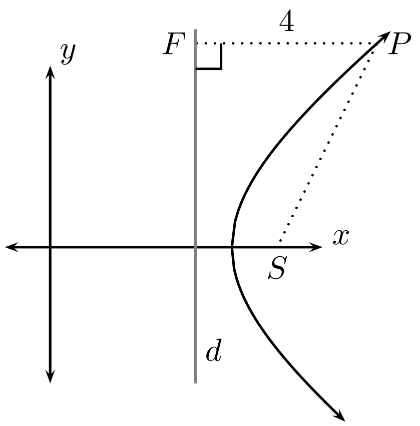
(d) (i) Find constants A, B and C such that **2**

$$\frac{x^2-x+1}{(x+1)^2} = A + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}\,.$$

(ii) Hence find $\displaystyle\int \frac{x^2-x+1}{(x+1)^2}\,dx$. **2**

(e) The polynomial $P(x) = 2x^3 - 3x^2 - 36x + k$ has a double zero. Find the possible values of k. **3**

(f)



The diagram above shows the right branch of the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$. The right directrix d and right focus S are shown. Let P be a fixed point on the curve and F a point on d, such that PF is perpendicular to d. It is known that $PF = 4$.

(i) Find the eccentricity of the hyperbola. **1**

(ii) Find the distance PS. **1**

Exam continues overleaf ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3 - i$ and $w = 2 + i$. Find the following in the form $x + iy$.

(i) $\overline{zw}$ **1**

(ii) $\left|\dfrac{z}{w}\right|$ **1**

(b) (i) Find the two square roots of $16 - 30i$. **2**

(ii) Hence solve $z^2 - 2z - (15 - 30i) = 0$. **1**

(c) In separate diagrams, sketch the region in the complex plane where:

(i) $|z - i| \le |z - 1|$ **2**

(ii) $0 < \arg\big(z - (1 + i)\big) \le \frac{\pi}{3}$ **2**

(d) (i) Write the complex number $1 + \sqrt{3}\,i$ in modulus-argument form. **1**

(ii) Use de Moivre's theorem to express $(1 + \sqrt{3}\,i)^5$ in the form $a + bi$. **2**

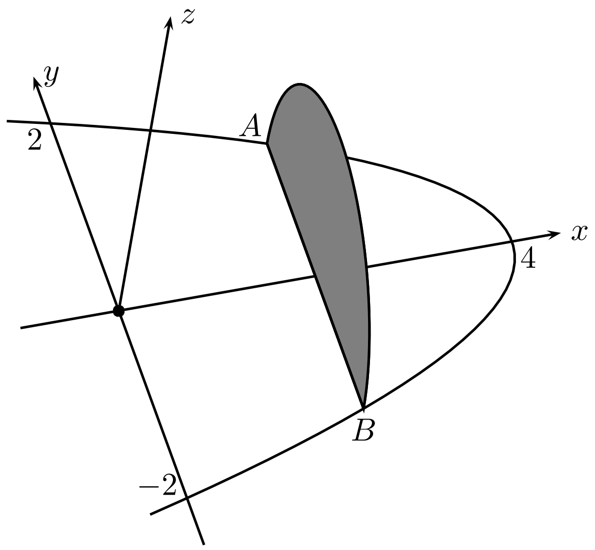
(e) The locus of a point P on the complex plane is defined by $|z - (1 + 2i)| = 3$.

(i) Sketch the locus of P. **2**

(ii) Find the maximum value of $|z|$. **1**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



The base of a solid is the region bounded by the parabola $y^2 = 4 - x$ and the y-axis, as in the diagram above. A typical vertical cross-section is a semi-circle parallel to the y-axis.

(i) Write an expression in terms of x for the area of a cross-section standing on the interval AB, as in the diagram. **1**

(ii) Find the volume of the solid. **2**

(b) Consider the polynomial equation $P(x) = 0$, where $P(x) = x^4 - 2x^3 + 6x^2 - 8x + 8$.

(i) Given that $x = 1 + i$ is a root of $P(x) = 0$, write down a second root. **1**

(ii) Write $P(x)$ as a product of two quadratic expressions. **2**

(iii) Fully factorise $P(x)$ over the complex numbers. **1**

(c) An object of mass 2 kilograms is projected vertically upwards from ground level at a speed of 20 m/s. It experiences a resistance of $\dfrac{v^2}{2}$ Newtons at a speed of v m/s, and reaches a maximum height H metres. Take upwards as positive and $g = 10$ m/s^2.

(i) Show that the acceleration is given by $\ddot{x} = \dfrac{-40 - v^2}{4}$. **1**

(ii) Show that the time it takes for the object to reach its maximum height is approximately 0·8 seconds. **2**

(iii) Find the maximum height H reached by the object. **2**

(iv) Calculate the speed of projection required to reach a maximum height of $2H$ metres. **3**

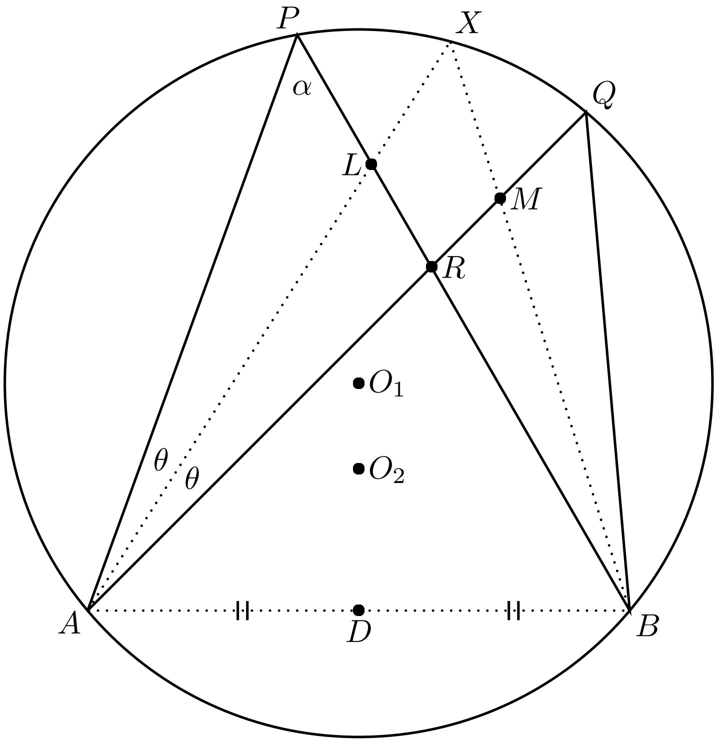
Exam continues overleaf ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Find the roots of the equation $z^5 - 1 = 0$. You may leave the complex roots in modulus–argument form. **2**

(ii) Hence find the exact value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$. **2**

(b) The diagram below is reproduced on a tear off sheet at the end of this paper. It should be handed in with your solution to this question.



The minor arc AB on the circle $\mathcal{C}_1$ subtends angles at P and Q on $\mathcal{C}_1$. The point X is chosen on the minor arc PQ such that AX bisects $\angle PAQ$. Suppose that AX and BX intersect BP and AQ at L and M respectively, and that AQ and BP intersect at R. Let O_1 be the centre of $\mathcal{C}_1$, D be the midpoint of AB, $\angle PAX = \theta$ and $\angle APB = \alpha$.

(i) Show that BX bisects $\angle PBQ$. **1**

(ii) Show that A, L, M and B lie on the circumference of a circle $\mathcal{C}_2$ and call its centre O_2. **1**

(iii) By considering the perpendicular bisector of the chord AB, explain why O_1, O_2 and D are collinear. **1**

(iv) Show that $\angle AO_2D = \alpha + \theta$. **1**

(v) Show that $\angle O_1AO_2 = \theta$. **1**

Exam continues next page ...

QUESTION FOURTEEN (Continued)

(c) Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b$, and let $P(a\cos\theta, b\sin\theta)$ be a point on the ellipse in the first quadrant.

Let $O(0,0)$ be the centre of the ellipse. Let T and N be the y-intercepts of the tangent and normal respectively to the ellipse at P. You may use the fact that the tangent at P has equation $ay\sin\theta + bx\cos\theta = ab$.

(i) Show the normal to the ellipse at P has equation **2**

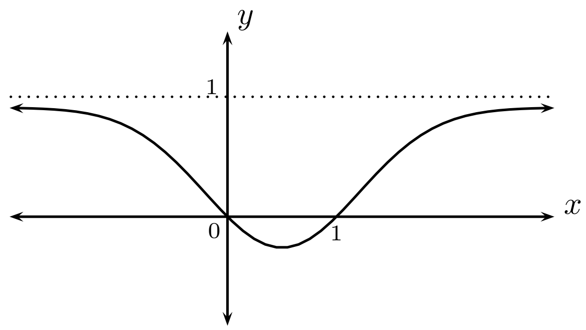
$$by\cos\theta - ax\sin\theta = (b^2 - a^2)\cos\theta\sin\theta.$$

(ii) Show that the product $TO \times ON$ is independent of θ. **2**

(iii) Suppose that the circle with diameter TN is drawn. Using the result in (ii), or otherwise, show that the points of intersection of the circle with the x-axis are independent of θ. **2**

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the graph $y = f(x)$ sketched below.



Draw separate one-third page sketches of the following graphs:

(i) $y = \big(f(x)\big)^2$ **2**

(ii) $y = \ln f(x)$ **2**

(iii) $y = xf(x)$ **2**

(b) Let $I_n = \displaystyle\int_0^1 x^n \sqrt{1 - x^3}\, dx$, for $n \geq 2$.

(i) By writing $x^n \sqrt{1 - x^3} = x^{n-2} \times x^2 \sqrt{1 - x^3}$, or otherwise, show that **3**

$$I_n = \frac{2n - 4}{2n + 5} \times I_{n-3}, \quad \text{for } n \geq 5$$

(ii) Hence find I_8. **1**

(c) Let $z = \cos\theta + i\sin\theta$.

(i) Show that $\sin n\theta = \dfrac{z^n - z^{-n}}{2i}$ and $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$. **2**

(ii) Hence, or otherwise, prove the identity **3**

$$32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2.$$

Exam continues next page ...

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The circle $x^2 + y^2 + ax + by + c = 0$ cuts the rectangular hyperbola $x = kt$, $y = k/t$ in four points P, Q, R and S defined by parameters p, q, r and s respectively.

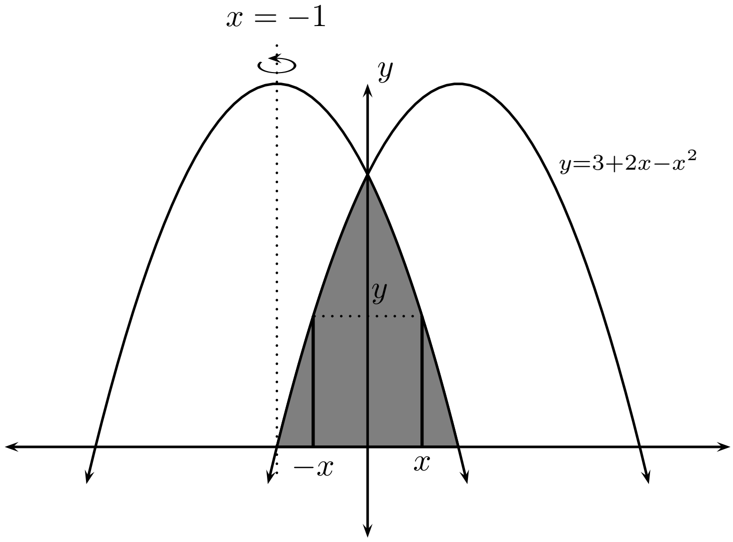
(i) Show that the parameters are roots of the quartic equation **1**

$$k^2t^4 + akt^3 + ct^2 + bkt + k^2 = 0.$$

(ii) Show that $pqrs = 1$. **1**

(iii) Show that if QS is a diameter of the hyperbola then PR is a diameter of the circle. **3**

(b)



The diagram above shows the parabola $y = 3 + 2x - x^2$ and its reflection in the y-axis. The vertical strips shown will generate cylindrical shells of the same height y when rotated about the line $x = -1$.

(i) Show that the sum of the areas of these two cylindrical shells is $4\pi y$. **1**

(ii) Find the volume of the solid formed when the region bounded by the parabolas and the x-axis is rotated about the line $x = -1$. **2**

(c) The polynomial equation $x^3 - px + 1 = 0$ has roots α, β and γ.

(i) Find the cubic equation whose roots are $\dfrac{1}{\alpha^3}$, $\dfrac{1}{\beta^3}$ and $\dfrac{1}{\gamma^3}$. **2**

(ii) Find an expression for $\dfrac{\beta\gamma}{\alpha^5} + \dfrac{\alpha\gamma}{\beta^5} + \dfrac{\alpha\beta}{\gamma^5}$ in terms of p. **2**

(d) Use the identity $(1 + x)^{2n+1}(1 - x)^{2n} = (1 + x)(1 - x^2)^{2n}$ to prove that **3**

$$\binom{2n+1}{0}\binom{2n}{0} - \binom{2n+1}{1}\binom{2n}{1} + \cdots + \binom{2n+1}{2n}\binom{2n}{2n} = (-1)^n\binom{2n}{n}.$$

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\ \ n \neq -1;\ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

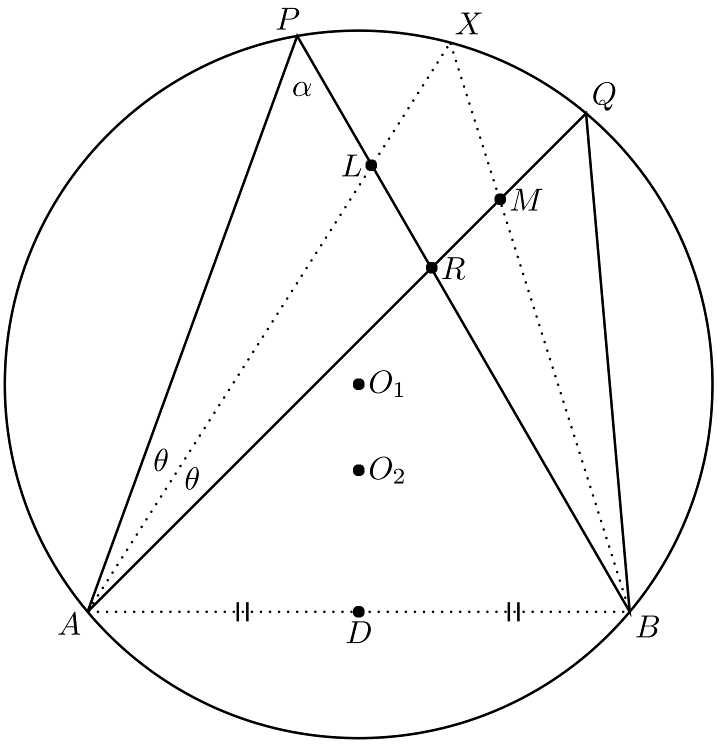
$$\text{NOTE}:\ \ln x = \log_e x,\ x > 0$$

Candidate number: ....

Detach this sheet and bundle it with the rest of question fourteen.

QUESTION FOURTEEN

(b) The circle geometry diagram from question 14b is reproduced below. Hand this in with your solution to question 14b.



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SYDNEY GRAMMAR SCHOOL

2012
Trial Examination
FORM VI
MATHEMATICS EXTENSION 2
Monday 6th August 2012

- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

2012 EXT. II S.G.S TRIAL HSC.

MULTIPLE CHOICE (1 mark each)

Q1

$$\int_0^{\pi} 5\sin x\cos^4 x\,dx = -\left[\cos^5 x\right]_0^{\pi}$$
$$= -\{-1-1\}$$
$$= 2 \quad \boxed{B}$$

Q2

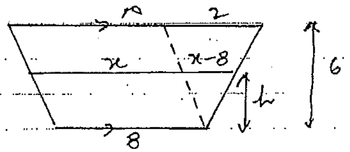
$S'(-ae,0)$ $S(ae,0)$

$d = 2ae$

$d = 2\times 10\times \frac{4}{5}$

$= 16 \quad \boxed{B}$

Q3



$\frac{x-8}{2} = \frac{h}{6}$

$x-8 = \frac{h}{3}$

$x = 8+\frac{h}{3} \quad \boxed{C}$

Q4

$\ddot{x} = 64 - \frac{v^2}{100}$

when $\ddot{x}=0$, $v = 80\text{ m/s}$

$80\text{ m/s} = 288\text{ km/h} \quad \boxed{A}$

Q5

$$V = \pi\int_0^4 (3-y)^2 - 1^2\,dx$$
$$= \pi\int_0^4 8 - 6\sqrt{x} + x\,dx \quad \boxed{B}$$

Q6

$\vec{BC} = z_3 - z_2$

$\vec{OD} = \vec{OA} + \vec{BC}$

$= z_1 + z_3 - z_2$

Now $\vec{DB} = \vec{OB} - \vec{OD}$

$= z_2 - (z_1 + z_3 - z_2)$

$= 2z_2 - z_1 - z_3 \quad \boxed{C}$

Q7

$z^3 + 1 = 0$

roots: ω, $\frac{1}{\omega}$, -1

sum: $\omega + \frac{1}{\omega} - 1 = 0$

$\omega^2 - \omega + 1 = 0 \quad \boxed{D}$

Q8

$$\lim_{N\to\infty}\int_0^N e^{-x}\,dx$$
$$= \lim_{N\to\infty}\left[-e^{-x}\right]_0^N$$
$$= \lim_{N\to\infty}\left(-\frac{1}{e^N}\right) - -1$$
$$= 1 \quad \boxed{B}$$

Q9

$y = f(x)$

$y = f(2-x)$

Reflect about the line $x=1$ $\boxed{A}$

Q10 The angle between the chord and tangent at the point of contact F is NOT equal to the angle in the alternate segment. ($\angle FBD \neq \angle DFE$)

$\boxed{D}$

(10)

QUESTION 11

(a)
$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + c \checkmark$$

(b)
$$\int \frac{x+1}{x-2}\,dx = \int \frac{x-2+3}{x-2}\,dx$$
$$= \int 1 + \frac{3}{x-2}\,dx \checkmark$$
$$= x + 3\ln(x-2) + c \checkmark$$

(c)
$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$$

let $x = 2\cos\theta$

$dx = -2\sin\theta\,d\theta$

when $x=1$, $\theta = \frac{\pi}{3}$ ✓

when $x=\sqrt{3}$, $\theta = \frac{\pi}{6}$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{-2\sin\theta\,d\theta}{4\cos^2\theta\sqrt{4-4\cos^2\theta}}$$
$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{-2\sin\theta\,d\theta}{8\cos^2\theta\sin\theta}$$
$$= -\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \sec^2\theta\,d\theta \checkmark$$
$$= -\frac{1}{4}\left[\tan\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{6}}$$
$$= -\frac{1}{4}\left\{\tan\frac{\pi}{6} - \tan\frac{\pi}{3}\right\}$$
$$= -\frac{1}{4}\left\{\frac{1}{\sqrt{3}} - \sqrt{3}\right\}$$
$$= -\frac{1}{4}\left(\frac{\sqrt{3}}{3} - \sqrt{3}\right) = \frac{\sqrt{3}}{6} \checkmark$$

(d)(i) $x^2 - x + 1 \equiv A(x+1)^2 + B(x+1) + C$

put $x=-1$: $C = 3$

put $x=0$: $A + B + 3 = 1$

$A + B = -2$ ✓ ①

put $x=1$: $4A + 2B + 3 = 1$

$2A + B = -1$ ②

② − ①: $A = 1$, ∴ $B = -3$

(ii) The integral becomes:

$$\int 1 - \frac{3}{x+1} + \frac{3}{(x+1)^2}\,dx$$
$$= x - 3\ln(x+1) - \frac{3}{(x+1)} + c \checkmark$$

(e) $P(x) = 2x^3 - 3x^2 - 36x + k$

$P'(x) = 6x^2 - 6x - 36$

$x^2 - x - 6 = 0$ ✓

$(x-3)(x+2) = 0$

$P(3) = P(-2) = 0$

$P(3)$: $54 - 27 - 108 + k = 0$

$k = 81$ ✓

or

$P(-2)$: $-16 - 12 + 72 + k = 0$

$k = -44$ ✓

(f)(i) $b^2 = a^2(e^2-1)$

$9 = 16(e^2-1)$

$\frac{9}{16} = e^2 - 1$

$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$

(15)

(ii) $\frac{PS}{PF} = e$ ✓

$\frac{PS}{4} = \frac{5}{4}$, ∴ $PS = 5$ units ✓

QUESTION 12

(a) $z = 3 - i \qquad w = 2 + i$

(i) $\overline{zw} = \overline{(3-i)(2+i)}$

$= \overline{7+i}$

$= 7 - i$ ✓

(ii) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{\sqrt{10}}{\sqrt{5}}$

$= \sqrt{2}$ ✓

(b)

(i) let $z = a + ib$

$(a+ib)^2 = 16 - 30i$

R: $a^2 - b^2 = 16$

Im: $2ab = -30$

$a = 5, \quad b = 3$ or

$a = -5, \quad b = 3$

$z = 5 - 3i$ ✓ or $z = -5 + 3i$ ✓

(ii) $z^2 - 2z - (15 - 30i) = 0$

$\Delta = b^2 - 4ac$

$= 4 + 4(15 - 30i)$

$= 4(1 + 15 - 30i)$

$= 4(16 - 30i)$

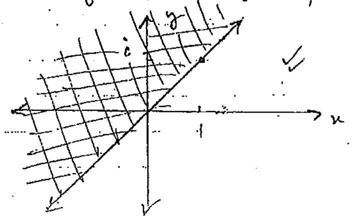
$= 2^2(5 - 3i)^2$ (i)

$z = \frac{+2 \pm 2(5 - 3i)}{2}$

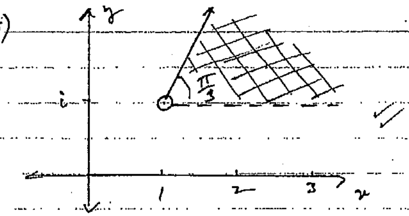
$z = 1 \pm (5 - 3i)$ ✓

$z = 6 - 3i$ or $z = -4 + 3i$

(c) (i) $|z - i| \leq |z - 1|$



(ii)



(d)(i) $1 + \sqrt{3}i = 2 \operatorname{cis} \frac{\pi}{3}$ ✓

(ii) $(1 + \sqrt{3}i)^5 = \left(2 \operatorname{cis} \frac{\pi}{3}\right)^5$

$= 32 \operatorname{cis} \frac{5\pi}{3}$ ✓

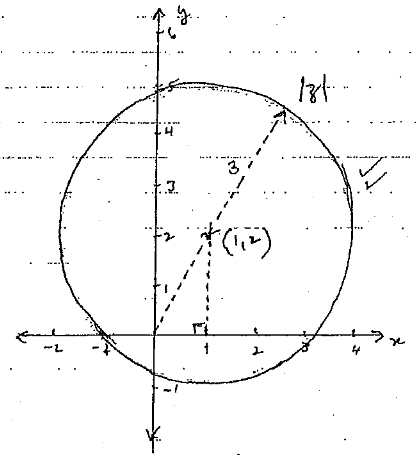
$= 32\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 16(1 - \sqrt{3}i)$ ✓

(e) $|z - (1 + 2i)| = 3$

$(x-1)^2 + (y-2)^2 = 9$

(i)



(ii) $|z|_{max} = 3 + \sqrt{5}$ ✓

(15)

QUESTION 13

(a)

(i) $A = \frac{\pi}{2} y^2$

$A = \frac{\pi}{2}(4 - x)$ ✓

(ii) $V = \frac{\pi}{2}\int_0^4 4 - x \, dx$

$V = \frac{\pi}{2}\left[4x - \frac{x^2}{2}\right]_0^4$ ✓

$V = 4\pi$ cubic units ✓

(b) $x^4 - 2x^3 + 6x^2 - 8x + 8 = 0$

(i) $1 - i$ is another root. ✓

(ii) $x^2 - Sx + P$

$x^2 - 2x + 2$ is one quadratic factor.

$$\begin{array}{r} x^2 + 4 \\ x^2 - 2x + 2 \,\big)\, x^4 - 2x^3 + 6x^2 - 8x + 8 \\ x^4 - 2x^3 + 2x^2 \\ \hline 4x^2 - 8x + 8 \end{array}$$ ✓

$P(x) = (x^2 - 2x + 2)(x^2 + 4)$ ✓

(iii) $P(x) = (x - 1 - i)(x - 1 + i) \times (x - 2i)(x + 2i)$ ✓

(c) (i) $m\ddot{x} = -mg - \frac{v^2}{2}$

$2\ddot{x} = -20 - \frac{v^2}{2}$ ✓

$\ddot{x} = \frac{-40 - v^2}{4}$

(ii) $\frac{dv}{dt} = -\frac{v^2 + 40}{4}$

$\frac{dt}{dv} = -\frac{4}{v^2 + 40}$

$t = \frac{-4}{2\sqrt{10}} \tan^{-1}\frac{v}{2\sqrt{10}} + c$ ✓

when $t = 0, \quad v = 20$

$c = \frac{2}{\sqrt{10}} \tan^{-1}\sqrt{10}$

$t = \frac{2}{\sqrt{10}} \tan^{-1}\sqrt{10} - \frac{2}{\sqrt{10}} \tan^{-1}\frac{v}{2\sqrt{10}}$ ✓

max height when $v = 0$:

$t = \frac{2}{\sqrt{10}} \tan^{-1}\sqrt{10}$

$t \doteq 0.8$ s

(iii) $\ddot{x} = \frac{-40 - v^2}{4}$

$v\frac{dv}{dx} = -\frac{v^2 + 40}{4}$

$\frac{dv}{dx} = -\frac{v^2 + 40}{4v}$

$\frac{dx}{dv} = -\frac{4v}{v^2 + 40}$ ✓

$x = -2\ln(v^2 + 40) + c$ —(i)

when $x = 0, \ v = 20$

$c = 2\ln(440)$

$x = 2\ln\left(\frac{440}{v^2 + 40}\right)$

max. height when $v = 0$

$x_{max} = 2\ln 11$ m. ✓//

(iv) $x = -2\ln(v^2 + 40) + c$

from (i).

when $x = 0$, put $v = V$

$x = 2\ln\left(\frac{V^2 + 40}{v^2 + 40}\right)$ ✓

when height = 2H
= 4 ln 11 m

$2\ln\left(\frac{v^2+40}{v^2+40}\right) = 4\ln 11$ ✓

$\frac{V^2+40}{v^2+40} = 121$

put $v = 0$ for max. height.

$V^2 + 40 = 40 \times 121$
$V^2 = 120 \times 40$
$V^2 = 4800$
$V = 40\sqrt{3}$ m/s ✓

(15)

QUESTION 14

(a) (i) $z^5 = 1$
let $z = \text{cis}\,\theta$
$(\text{cis}\,\theta)^5 = 1$
Using de Moivre's thm:
$\text{cis}\,5\theta = 1$
Re: $\cos 5\theta = 1$
$5\theta = 2\pi n$ ✓
$\theta = \frac{2\pi n}{5}$, $n = 0, \pm1, \pm2$

Roots are:
1, $\text{cis}\left(\frac{2\pi}{5}\right)$, $\text{cis}\left(-\frac{2\pi}{5}\right)$, $\text{cis}\left(\frac{4\pi}{5}\right)$ and $\text{cis}\left(-\frac{4\pi}{5}\right)$ ✓

(ii) $\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) = 2\cos\frac{2\pi}{5}$
$\text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) = 2\cos\frac{4\pi}{5}$

Sum of roots ✓ $= -\frac{b}{a} = 0$

$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} + 1 = 0$
$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ ✓

(b)(i) $\angle PAX = \angle PBX = \theta$
$\angle XAQ = \angle XBQ = \theta$
(angles drawn to the circumference standing on the same arc are equal.) $\angle PBX = \angle XBQ$
$\therefore BX$ bisects $\angle PBQ$. ✓

(ii) $\angle LAR = \theta$ (given)
$\angle MBR = \theta$ (i)
Angles in the alternate segment are equal, hence ALMB are concyclic points and lie on the circumference of C_2. ✓

(iii) Since AB is a common chord of circles C_1 and C_2, the perpendicular bisector of chord AB passes through the centres O_1 and O_2. Hence O_1, O_2 and D are collinear. ✓

(iv) $\angle ALR = \theta + \alpha$
(exterior angle of triangle thm.)
$\angle AO_2B = 2(\theta + \alpha)$
(angle at the centre is twice the angle drawn to the circumference of C_2 standing on the arc AB.)
$\triangle AO_2D \equiv \triangle DO_2B$ (SSS)
$\angle AO_2D = \angle BO_2D$ ✓
(corresponding angles of cong. triangle
$\angle AO_2D + \angle BO_2D = 2(\theta+\alpha)$
$2\angle AO_2D = 2(\theta+\alpha)$
$\therefore \angle AO_2D = \theta + \alpha$

(v) $\angle AO_1B = 2\alpha$ (angle drawn to the centre is twice the angle drawn to the circumference.)
$\angle AO_1D = \angle BO_1D$
(matching angles of cong. triangles,
$\angle AO_1D = \alpha$, but
$\angle AO_2D = \alpha + \theta$ (part iv)
$\therefore \angle O_1AO_2 = \theta$ (external angle of triangle theorem. ✓

... cont.

(c)

(i) $x = a\cos\theta$ $\quad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$ $\quad \frac{dy}{d\theta} = b\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= -\frac{b}{a}\,\frac{\cos\theta}{\sin\theta}$

gradient of the normal at $P(a\cos\theta, b\sin\theta)$ is

$m = \frac{a}{b}\,\frac{\sin\theta}{\cos\theta}$ Using ✓

$y - y_1 = m(x - x_1)$

$y - b\sin\theta = \frac{a}{b}\,\frac{\sin\theta}{\cos\theta}(x - a\cos\theta)$

$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

$by\cos\theta - ax\sin\theta = (b^2 - a^2)\sin\theta\cos\theta$ ✓

(ii) $ay\sin\theta + bx\cos\theta = ab$

To find T put $x = 0$.

$y = \frac{b}{\sin\theta}$

$T\left(0, \frac{b}{\sin\theta}\right)$

$by\cos\theta - ax\sin\theta = (b^2 - a^2)\sin\theta\cos\theta$

To find N put $x = 0$. ✓

$N\left(0, \frac{b^2 - a^2}{b}\sin\theta\right)$

$TO = \frac{b}{\sin\theta}$

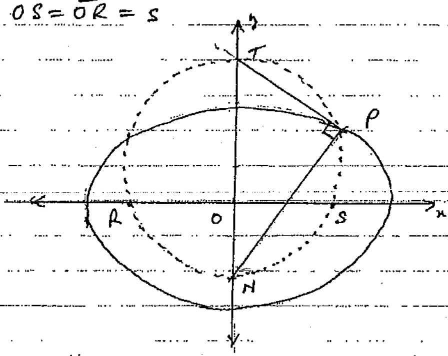
$ON = -\frac{(b^2 - a^2)}{b}\sin\theta$ (y co-ordinate of N is negative.)

$TO \times ON = \frac{b}{\sin\theta} \times \frac{(a^2 - b^2)}{b}\sin\theta$

$= a^2 - b^2$ ✓

(independent of θ).

(iii) $\angle TPN = 90°$, hence TN is a diameter of the circle. (angle in a semi circle.) Let $S(s, 0)$ and $R(-s, 0)$ be the x-intercepts of the circle. $\overline{OS} = \overline{OR} = s$



Using the intercepts of chords theorem:

$RO \times OS = TO \times ON$

$s^2 = a^2 - b^2$

$s = \sqrt{a^2 - b^2}$

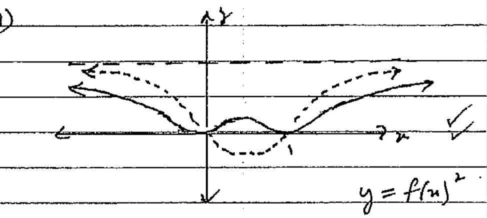
hence:

$R(-\sqrt{a^2-b^2}, 0)$ and $S(\sqrt{a^2-b^2}, 0)$ both co-ordinates are independent of θ.

(15)

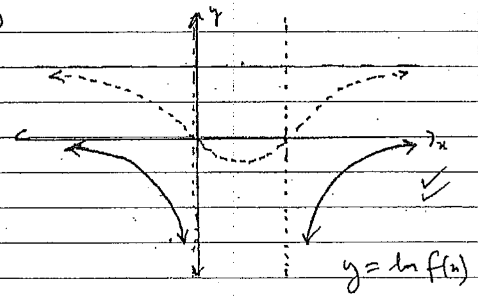
QUESTION 15

(a)



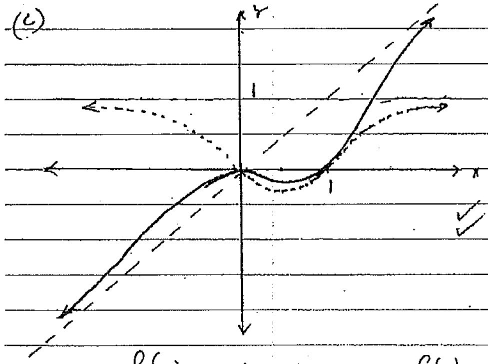
$y = f(x)^2$

(b)



$y = \ln f(x)$

(c)



as $f(x) \to 1$, $y = x f(x)$

$\to y = x$.

(b)

(i) $I_n = \int_0^1 x^n\sqrt{1 - x^3}\,dx$

$n \geq 2$.

$I_n = \int_0^1 x^{n-2} \times x^2\sqrt{1 - x^3}\,dx$

let $u = x^{n-2}$

$u' = (n-2)x^{n-3}$

$v' = x^2(1 - x^3)^{1/2}$

$v = -\frac{2}{9}(1 - x^3)^{3/2}$

$I_n = \left[uv\right]_0^1 + \frac{2}{9}(n-2)\int_0^1 x^{n-3}(1 - x^3)^{3/2}\,dx$

$I_n = \frac{2}{9}(n-2)\int_0^1 x^{n-3}(1 - x^3)^{3/2}\,dx$ ✓

$I_n = \frac{2}{9}(n-2)\int_0^1 x^{n-3}\sqrt{1 - x^3}\,(1 - x^3)\,dx$

$I_n = \frac{2}{9}(n-2)\left\{\int_0^1 x^{n-3}\sqrt{1 - x^3}\,dx - \int_0^1 x^n\sqrt{1 - x^3}\,dx\right\}$

$I_n = \frac{2}{9}(n-2)\left\{I_{n-3} - I_n\right\}$ ✓

$I_n\left\{1 + \frac{2}{9}(n-2)\right\} = \frac{2}{9}(n-2)\,I_{n-3}$

$I_n(9 + 2n - 4) = 2(n-2)\,I_{n-3}$

$I_n = \frac{2n-4}{2n+5}\,I_{n-3}$ ✓

(ii) $I_2 = \int_0^1 x^2\sqrt{1 - x^3}\,dx$

$= -\frac{2}{9}\left[(1 - x^3)^{3/2}\right]_0^1$

$= \frac{2}{9}$

$I_5 = \frac{10-4}{10+5} \cdot I_2 = \frac{6}{15} \times \frac{2}{9}$

$I_8 = \frac{16-4}{16+5}\,I_5 = \frac{12}{21} \times \frac{6}{15} \times \frac{2}{9}$

$= \frac{16}{315}$ ✓

(c)

(i) $z = \cos\theta + i\sin\theta$

① $z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ ✓

② $z^{-n} = \cos n\theta - i\sin n\theta$

①+②: $z^n + z^{-n} = 2\cos n\theta$

$\cos n\theta = \frac{z^n + z^{-n}}{2}$

①−②: $z^n - z^{-n} = 2i\sin n\theta$ ✓

$\sin n\theta = \frac{z^n - z^{-n}}{2i}$

(ii)

LHS $= 32\sin^4\theta\cos^2\theta$

$= 32\left(\frac{z - z^{-1}}{2i}\right)^4\left(\frac{z + z^{-1}}{2}\right)^2$

$= \frac{1}{2}\left(z - \frac{1}{z}\right)^4\left(z + \frac{1}{z}\right)^2$ ✓

$= \frac{1}{2}\left[\left(z - \frac{1}{z}\right)\left(z + \frac{1}{z}\right)\right]^2\left(z - \frac{1}{z}\right)^2$

$= \frac{1}{2}\left(z^2 - \frac{1}{z^2}\right)^2\left(z^2 - 2 + \frac{1}{z^2}\right)$ ✓

$= \frac{1}{2}\left\{\left(z^4 - 2 + \frac{1}{z^4}\right)\left(z^2 - 2 + \frac{1}{z^2}\right)\right\}$

$= \frac{1}{2}\left\{z^6 - 2z^4 + z^2 - 2z^2 + 4 - \frac{2}{z^2} + \frac{1}{z^2} - \frac{2}{z^4} + \frac{1}{z^6}\right\}$

$= \frac{1}{2}\left\{\left(z^6 + \frac{1}{z^6}\right) - 2\left(z^4 + \frac{1}{z^4}\right) - \left(z^2 + \frac{1}{z^2}\right) + 4\right\}$ ✓

$= \frac{1}{2}\left\{2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4\right\}$

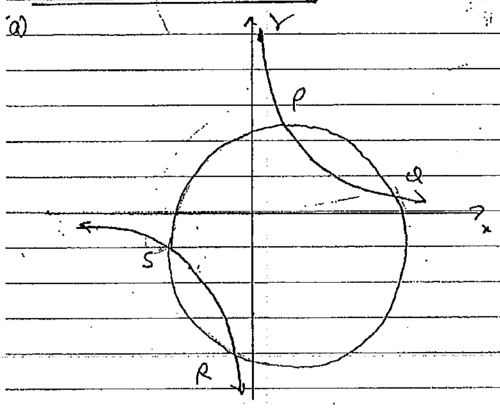
$= \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

$=$ RHS //

(15)

QUESTION 16

(a)



(i) $x = kt$ and $y = \frac{k}{t}$

$x^2 + y^2 + ax + by + c = 0$

$\therefore k^2t^2 + \frac{k^2}{t^2} + akt + \frac{bk}{t} + c = 0$

$k^2t^4 + k^2 + akt^3 + bkt + ct^2 = 0$

$k^2t^4 + akt^3 + ct^2 + bkt + k^2 = 0$

all points $P(pt, \frac{p}{t})$, Q, R ✓ and S satisfy this equation.

(ii) Product of the roots $= \frac{e}{a}$

$pqrs = \frac{k^2}{k^2}$ ✓

$pqrs = 1$

(iii) If QS is a diameter of the hyperbola,

$Q(qt, \frac{q}{t})$ $S(st, \frac{s}{t})$

then: $qt = -st$ and $\frac{q}{t} = -\frac{s}{t}$

hence $q = -s$ ✓

Now gradient of $P(kp, \frac{k}{p})$ $S(ks, \frac{k}{s})$:

$M_{PS} = \frac{k/s - k/p}{ks - kp}$

$= \frac{p-s}{sp(s-p)}$ ✓

$= -\frac{1}{sp}$

Similarly $M_{RS} = -\frac{1}{rs}$

$M_{PS} \times M_{RS} = \frac{1}{prs^2}$ Since $(s = -q)$

put $= \frac{1}{prs(-q)}$

$= -\frac{1}{prsq}$ ✓ $(prsq = 1)$

$= -1$

$\therefore \angle PSR = 90°$ and PR is a diameter. (angle in a semi-circle.)

(b)(i) $A_T = 2\pi Ry + 2\pi ry$

$R = x+1$ and $r = 1-x$

$A_T = 2\pi y\{x + 1 + 1 - x\}$ ✓

$A_T = 2\pi y \times 2$

$A_T = 4\pi y$ (as required)

(ii) $V = 4\pi\int_0^1 3 - 2x - x^2\,dx$

$V = 4\pi\left[3x - x^2 - \frac{x^3}{3}\right]_0^1$ ✓

$V = 4\pi\left[\left(3 - 1 - \frac{1}{3}\right) - (0)\right]$

$V = \frac{20\pi}{3}$ cubic units. ✓

Q16 ...

(c) $x^3 - px + 1 = 0$

(i) $\left(\frac{1}{x^{1/3}}\right)^3 - p\left(\frac{1}{x^{1/3}}\right) + 1 = 0$ ✓

$\frac{1}{x} + 1 = \frac{p}{x^{1/3}}$

$\left(\frac{1}{x} + 1\right)^3 = \frac{p^3}{x}$

$\frac{1}{x^3} + \frac{3}{x^2} + \frac{3}{x} + 1 = \frac{p^3}{x}$

$1 + 3x + 3x^2 + x^3 = p^3x^2$

$x^3 + (3 - p^3)x^2 + 3x + 1 = 0$ ✓

(ii) $x^3 - px + 1 = 0$

$\alpha\beta\gamma = -1$

$\frac{\beta\gamma}{\alpha^5} + \frac{\alpha\gamma}{\beta^5} + \frac{\alpha\beta}{\gamma^5}$

$= \alpha\beta\gamma\left\{\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}\right\}$

$= \alpha\beta\gamma\left\{\left(\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3}\right)^2 - 2\left\{\frac{1}{\alpha^3\beta^3} + \frac{1}{\alpha^3\gamma^3} + \frac{1}{\gamma^3\beta^3}\right\}\right\}$ ✓

$= -\left\{(p^3 - 3)^2 - 2 \times 3\right\}$

$= 6 - (p^6 - 6p^3 + 9)$

$= 6p^3 - p^6 - 3$ ✓

P.T.O.

Q16 (d) →

$LHS = (1+x)^{2n+1}(1-x)^{2n}$

$= \left\{\binom{2n+1}{0} + \binom{2n+1}{0}x + \cdots + \binom{2n+1}{2n-1}x^{2n-1} + \binom{2n+1}{2n}x^{2n} + \binom{2n+1}{2n+1}x^{2n+1}\right\}$

$\times \left\{\binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 + \cdots + \binom{2n}{2n-2}x^{2n-2} - \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}\right\}$ ✓

Consider the coefficient of x^{2n} in the product:

$= \binom{2n+1}{0}\binom{2n}{2n} - \binom{2n+1}{0}\binom{2n}{2n-1} + \cdots + \binom{2n+1}{2n}\binom{2n}{0}$

Using $\binom{n}{r} = \binom{n}{n-r}$: ✓

Coefficient of x^{2n} $= \binom{2n+1}{0}\binom{2n}{0} - \binom{2n+1}{0}\binom{2n}{1} + \cdots \binom{2n+1}{2n}\binom{2n}{2n}$

Now RHS $= (1+x)(1-x^2)^{2n}$

$= (1+x)\left\{\binom{2n}{0} - \binom{2n}{1}x^2 + \binom{2n}{2}x^4 + \cdots + (-1)^n\binom{2n}{n}x^{2n} + \cdots \binom{2n}{2n}x^{4n}\right\}$

No term in x^{2n} will be generated by the product of x in $(1+x)$ and the expansion since terms in the expansion have even powers. ✓

Coefficient of x^{2n}: $(-1)^n\binom{2n}{n}$

Hence

$\binom{2n+1}{0}\binom{2n}{0} - \binom{2n+1}{0}\binom{2n}{1} + \cdots \binom{2n+1}{2n}\binom{2n}{2n} = (-1)^n\binom{2n}{n}$.

(15)